1.1 Counting FUNdamentals!

1.1.1 Investigate: A Principled Way of Counting

Focus Question: What is it about the situation that determines which mathematical operation(s) to use?

In Case You Are Stuck…

- Try writing down some examples (and non-examples) of what you are trying to count.
- Try solving a smaller problem.
- Try counting the things you DON'T want, then use that to find how many things you DO want.

For each problem, work in a small group. Before trying to solve the entire problem, each group member should write down one example of an outcome you are being asked to count. Share it with your group to make sure you are trying to count the same thing. Then begin working on your own for a few minutes before sharing your progress with groupmates.

1. A restaurant offers 2 different choices of appetizer, 4 different choices of entree and 3 different choices of dessert.

   a. The restaurant begins offering a "Three in One" menu consisting of all three courses. Assuming all choices are available, how many different meals can a customer order from the "Three in One" menu? Explain how you know.

   b. A customer can also order a "Take Two" meal consisting of only two (out of the three) courses. Assuming all choices are available, how many different "Take Two" meals are possible? Explain how you know.

   c. Miguel eats here every day for lunch. If the restaurant decides to offer only their "Three in One" menu and "Take Two" menu for lunch, how many consecutive days can he eat here without ordering the same meal twice?

2. You have four different markers and seven different pencils.

   a. How many different sets of writing utensils can you create containing exactly one writing utensil, either a marker or a pencil?

   b. How many different sets of writing utensils can you create containing exactly two writing utensils, one marker and one pencil?

   c. How many different sets can you create containing any two different writing utensils?

   d. Suppose the markers are identical, and the pencils are identical. How many different sets of writing utensils can you make that have at least one writing utensil? Describe at least two different kinds of sets.

   e. Suppose you have ***m*** number of different markers and ***p*** number of different pencils. Please answer the questions from parts a-c again. Explain why you choose certain operation(s) to solve the problems. (Try to repeat the reasoning, not just substituting in numbers.)

3. Uriel is a student taking Discrete Math. All of the quizzes and tests are multiple choice, with four choices per question, and each question has only one correct answer.

   a. On Monday, Uriel's quiz has two multiple-choice questions. In how many different ways could Uriel complete this quiz if he answers all of the questions?

   b. On Tuesday, Uriel's quiz has three multiple-choice questions. In how many different ways could Uriel complete this quiz if he answers all of the questions?

   c. On Friday, Uriel is given a unit test with fifteen multiple-choice questions. In how many different ways could Uriel complete this test if he answers all of the questions?

   d. Determine the number of ways Uriel could complete the quiz for any number of questions (if there are still four choices per question).

   e. The following week, an exit slip was given with only four true/false questions. After he handed in his paper, Uriel realized that he forgot to answer one of the questions but wasn't sure which one. How many ways could he have completed his exit slip?

1.1.2 Reflect

1. In the above investigation, sometimes you had to add and other times you had to multiply. Explain when and why you would use each operation.

2. Suppose there are 3 freshmen and 4 sophomores in a class.

   Match each question on the left to an expression on the right that can be used to solve the problem. Explain your thinking.

Questions	Expressions
A. If one student from each grade level is selected to represent the class at a conference, how many different possibilities are there?	i. $3 + 4$
B. If one student is selected to represent the class at a meeting, how many different possibilities are there?	ii. $3 \cdot 4$
C. For four days in a row, a freshman is randomly chosen to read the class bulletin. How many different schedules of bulletin readers are possible?	iii. 3^4
	iv. 4^3

3. What questions or concerns do you still have about these counting problems?

1.1.3 Investigate: Don't Overcount!

Focus Questions:

- **How do you know you have not overcounted or undercounted?**
- **How do you correct for overcounting?**

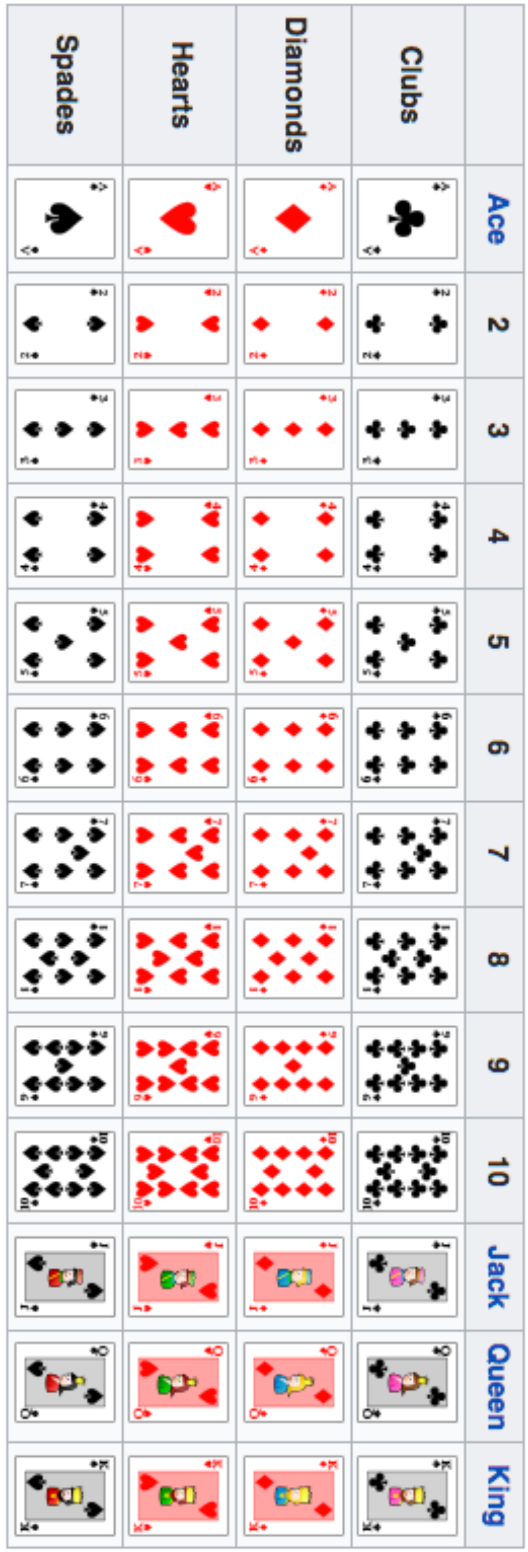
In Case You are Stuck…

- Try writing down some examples (and non-examples) of what you are trying to count.
- Try to create an organized way to count.
- Try solving a smaller problem.
- Sometimes it helps to think about what you DON'T want to count.
- When you work in groups or as a class, you may find that others approached the problem differently. When this happens, try to find how your approach relates to theirs.

In a standard 52-card deck there are four suits (hearts ♥, diamonds ♦, spades ♠, and clubs ♣), with 13 cards per suit. Each suit consists of card numbers ranging from 2 to 10, plus three face cards (jack, queen, and king) and an ace.

1. How many ways are there to select two different cards (one at a time without replacing the first card) from a standard 52-card deck such that the first card is a diamond and the second card is a heart?

Resource page for 1.2.3 Investigate: Don't Overcount!



Example set of 52 playing cards; 13 of each suit clubs, diamonds, hearts, and spades

2. How many ways are there to select two different cards (one at a time without replacing the first card) from a standard 52-card deck such that the first card is a diamond and the second card is an Ace? Which of these is a correct answer? Explain the reason for your choice(s).

    i. 13 + 4
    ii. 13 x 4
    iii. (13 x 4) – 1
    iv. (12 x 4) + (1 x 3)

3. A three-digit code is required to set this lock. Each digit is from 0-9. Determine the number of codes that contain at least 1 five.

http://www.ghlocks.com/photo/pl7899083_digit_luggage_combination_lock_travel_bag_combination_lock.jpg

1.1.4 Reflect

In Investigation 1.2.1, we asked the focus question, "**What is it about the situation that determines which mathematical operation(s) to use?**" You probably used multiplication and addition at different points as you solved all the problems. But, how can you tell when to use each one?

Consider the following situation.

Trang was asked to solve the following problems.

I. In a standard deck of cards, how many cards are a diamond or a king?

II. In a standard deck of cards, how many cards are a diamond and a king?

Her friend had told her a trick, "If the problem has the word 'or' then you add. If it has the word 'and' then you multiply." Trang used the trick to reason, "Since there are 13 diamond cards and 4 Kings, the number of possibilities for the first problem is 13 + 4 = 17, and the number of possibilities for the second problem is 13 x 4 = 52." Explain why Trang's reasoning is incorrect.

1.1.5 Formalize: Counting Principles

"What is it about the situation that determines which mathematical operation(s) to use?" In section 1.2.4 you learned why identifying key words like "and" and "or" might not help, but what does?

In Investigation 1.2.1, problem #1, you solved the following problem by just thinking about it.

A restaurant offers 2 different choices of appetizer, 4 different choices of entree and 3 different choices of dessert. A customer can order a "Take Two" meal consisting of only two (out of the three) courses. Assuming all choices are available, how many different "Take Two" meals are possible?

Along the way, you probably used multiplication and addition at different times to solve the problem. Take a moment now to solve it again thinking about when and why you chose to add or multiply. Then read the following notes, considering your solution and identify when and why you used the addition and multiplication by using the guiding principles defined below. To demonstrate the fact that there may be multiple ways to think about these principles, we present two formulations of the Addition Principle and one for the Multiplication Principle.

The Addition Principle

The Addition Principle (Formulation 1, Adapted from Rosen (1999)): *If a first task can be done in **n** ways and a second task in **m** ways, and if these tasks cannot occur at the same time, then there are **n + m** ways to do either task.*
(Note this can be generalized to more than two tasks.)

The Addition Principle (Formulation 2, Adapted from Tucker (2002)): *If there are **n** different objects in the first set and **m** different objects in a second set, and if the different sets have no elements in common, then the number of ways to select an object from one of the sets is **n + m**.*
(Note this can be generalized to more than two tasks.)

Example: Consider the problem "Suppose there are 10 puppies and 15 kittens at the animal shelter, and I want to select only one animal to take home with me. How many different possibilities are there for which animal I can take?" The answer is 10 + 15.

In terms of Formulation 1, we can think of the first task as choosing a puppy (with 10 ways to complete that task) and the second task as choosing a kitten (with 15 ways to complete that task). These tasks cannot occur at the same time because I am only taking one animal (that is, either one or the other can occur). Thus, there are 10 + 15 ways to complete either task.

In terms of Formulation 2, we can think of there being a set of puppies and a set of kittens. The sets have no elements in common, as no puppy can also be a kitten. The number of ways to select a puppy from the set of puppies is 10, and the number of ways to select a kitten from the set of kittens is 15. Thus, the number of ways to select an object from one of the sets is 10 + 15.

The Multiplication Principle

*If a procedure can be broken into two stages, and if there are **N** outcomes in the first stage and **M** outcomes in the second stage (independent of the choice in the first stage), then the total procedure has **N· M** composite outcomes.*

(Note this can be generalized to more than two stages.)

Example: Consider the problem "Suppose there are 10 puppies and 15 kittens and the animal shelter, and I want to select one of each animal to take home with me. How many different possibilities are there for which two animals I can take?" Here the answer is 10 x 15.

We can think of the overall procedure as choosing two animals to bring home, and the two stages are picking a puppy and picking a kitten, respectively. There are 10 outcomes of the first stage (the 10 possible puppies), and there are 15 outcomes of the second stage (the 15 possible kittens). Notice that the number of outcomes of the second stage is independent of the choice in the first stage – that is, no matter which puppy I pick I *always* have 15 kittens to pick. Because any of the 10 puppies can be paired with any of the 15 kittens, there are 10 x 15 total composite outcomes (where the composite outcomes are puppy-kitten pairs).

Can you answer the question now? "**What is it about the situation that determines which mathematical operation(s) to use?**"

Revisit your solution to this problem in light of what you have just read pointing out when and why you used addition and multiplication.

A restaurant offers 2 different choices of appetizer, 4 different choices of entree and 3 different choices of dessert. A customer can order a "Take Two" meal consisting of only two (out of the three) courses. Assuming all choices are available, how many different "Take Two" meals are possible?

Counting FUNdamentals! - Problem Set

Check for Understanding

1. A food stand sells three different types of food items: drinks, entrées, and sides. For drinks, customers can either have water, milk or soda; for the entrées they can have pizza, hamburgers, or chicken nuggets, and for the side the choices are chips or fruit.

   a. A lunch combo contains a drink, an entree, and a side. List all the possible lunch combos you could order.

   b. Andy frequently eats at this food stand. He buys either just one type, two different types, or all three types. He wonders how many different purchases are possible, assuming all choices are available and only one item is bought from each type. Help Andy figure this out.

   c. In response to Andy's question in part (b), Bob says, *"For a drink, Andy has 4 different choices, either water, milk, soda, or nothing. Similarly, for an entrée, he also has 4 different choices, either pizza, hamburger, chicken nuggets, or nothing. Each choice of drink can be paired with a choice of entrée, so Andy has 4 x 4 or 16 choices for drink and entrée. Each of these 16 choices can be paired with one of three choices for a side, either chips, fruits, or nothing. Therefore, the total number of different purchases Andy can have is 4 x 4 x 3 = 48."* Do you agree with Bob's explanation? Why or why not?

2. In the upcoming election for class council, students can choose to elect Ariana or Barbara for President, Carlos or Diego for Vice President, and Ellie, Francisco, or Georgiana for Treasurer.

   a. Draw a diagram illustrating the possible ways a student can complete a ballot, if they must cast a vote for every office.

   b. How many different ways can a ballot be completed if a student must cast a vote for each office?

   c. How many different ways can a ballot be completed if a student can choose not to vote in some (or all) of the elections?

Repeated Reasoning

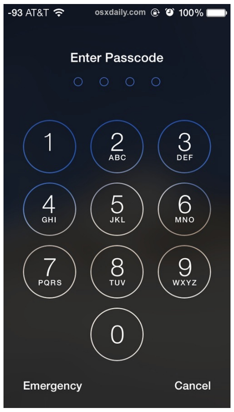
3. In a basketball league, there are 3 teams from the eastern region, 4 teams from the western region and 2 teams from the central region.
   a. Each region needs to select one team to attend a union meeting. How many ways can they do this
   b. If one team is to be selected for an award, how many different selections are possible?
   c. If two teams from two different regions are selected to play to raise funds for a charity, how many ways are there to select them?

4. My brother has 4 shirts, 5 pairs of pants, and 7 hats.
   a. He wants to pick an outfit consisting of a shirt and a pair of pants to wear to school. In how many ways can he do this?
   b. My brother's sometimes lazy. He'll throw on two items of clothing (where an item is either a shirt, a pair of pants, or a hat) and then try and leave the house. How many ways can he do this?
   c. Of course, if he tries to leave the house without putting on pants, our mom will insist that he puts on an outfit consisting of a shirt, a pair of pants, and a hat. How many ways can he put on such an outfit?

5. Your book shelf consists of 8 novels and 5 nonfictions. Give an example of a question for which the answer is:
   a. 13
   b. 40
   c. 80
   d. 156
   e. 169

6. A multiple-choice quiz contains 3 questions and each of these has 3 answer choices (A, B, or C), with one correct answer per question. The teacher allowed her students to skip one of the questions if they wanted to. After class, Abby and Bella talked to each other about their quiz. Abby said, "I answered all the questions, there were 27 different ways I could have completed the quiz." Bella commented, "I skipped one of the questions. There were also 27 different ways that I could have completed the quiz."
   a. How is it that the two students have the same number of ways to complete the quiz? Are they counting the same thing? Why or why not?
   b. How many ways can students answer the quiz?

7. An iPhone passcode contains 4 digits, each digit is selected from 0-9. How many passcodes contain at least one 7?



8. The people of Zedland have a strange language. Their language comprises of every possible "word" (formed as a sequence of English letters) that is three letters long and contains at least one Z. How many words do they have in their language? Solve this problem in at least two ways.

Diving Deeper

9. License plates in Massachusetts have six characters. These characters can either be digits (from 0-9) or letters in the English alphabet.

   a. How many six-character plates contain at least one number?

   b. How many six-character plates contain the letter X somewhere in the first three characters (and the last three characters may or may not contain the letter X)?

   c. How many six-character plates contain the letter X or the number 0, and possibly both?

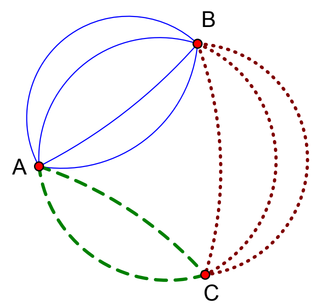
   d. How many six-character plates that contain only letters have vowels appearing only as the first or last letter, and possibly both?

   e. Your classmates come up with the following answers to the previous question. How might they have gotten those answers? Did they count correctly? If not, explain to them why they counted wrong.

      i. $5^2 \cdot 21^4$
      ii. $5^2 \cdot 26^4$
      iii. $21^2 \cdot 26^4$
      iv. $26^6 - 21^2 \cdot 26^4$

10. Eight members of a student club are lining up in a row to take a photo, but two of them, Alice and Bob, refuse to stand next to each other. How many ways can the students arrange themselves so that Alice and Bob are not next to each other?

11. There are four different roads between town A and town B, three different roads between town B and town C, and two different roads between town A and town C.



  a. How many different routes are there from A to C, if each city can only be visited once?
  b. How many different routes are there from A to C and back if each city can only be visited once in each direction?
  c. How many routes are there from A back to A, if each of B and C are visited exactly once?

12. It's cold outside, so I want to wear gloves. Unfortunately, my glove drawer is a mess – I have 3 black left gloves, 4 black right gloves, 5 blue left gloves, and 6 blue right gloves.
  a. In how many ways can I pick a left glove and a right glove of the same color to wear?
  b. In how many ways can I pick a left glove and a right glove to wear if I don't care about the colors?
  c. In how many ways can I pick two gloves of different colors to wear if I don't care about having two left gloves or two right gloves?
  d. It's dark outside, so I can't see. What is the minimum number of gloves I need to grab to ensure that I have at least one left glove and at least one right glove? Explain your reasoning.

13. There are 24 people in the Combinatorics Club. 10 of them like classical music. 16 of them like rock music. 5 of them don't like either. How many members like classical music, but dislike rock music?

14. How many integers between 1 and 100 inclusive are divisible by 2 or 3?
15. How many four-digit positive integers have at least one digit that is a 3 or 4?

16. How many sequences of length four can be formed using the digits 0, 1, 2, 3, 4, 5, 6 so that exactly two distinct digits appear? For example, 0002 and 0440 are such sequences while 0224 and 2222 are not.