1.1 The Legend of the Golden Disks

The Tower of Hanoi, also known as the Tower of Brahma, is a puzzle invented by E. Lucas in 1883. According to legend, in an Indian temple that contains a large room with three poles surrounded by 64 golden disks, the priests of Brahma have been moving these golden disks, in accordance with the rules of the puzzle. Based on an ancient prophecy, when the last move of the puzzle is completed, the world will end. The puzzle has appeared many times in popular culture. In the film Rise of the Planet of the Apes in 2011, it appeared as an intelligence test for apes under the name "Lucas Tower."

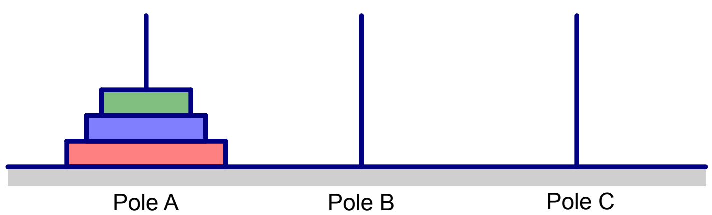
1.1.1 Investigate: Keep It Down!

Focus question: What is the minimum number of moves needed to solve the puzzle?

You are given a stack of disks arranged from largest on the bottom to smallest on top, placed on a pole. The goal is to move the stack of disks from the left-most pole (A) to another pole with the following conditions:

- Only one disk may be moved at a time.
- A larger disk may <u>not</u> be put on top of a smaller disk.

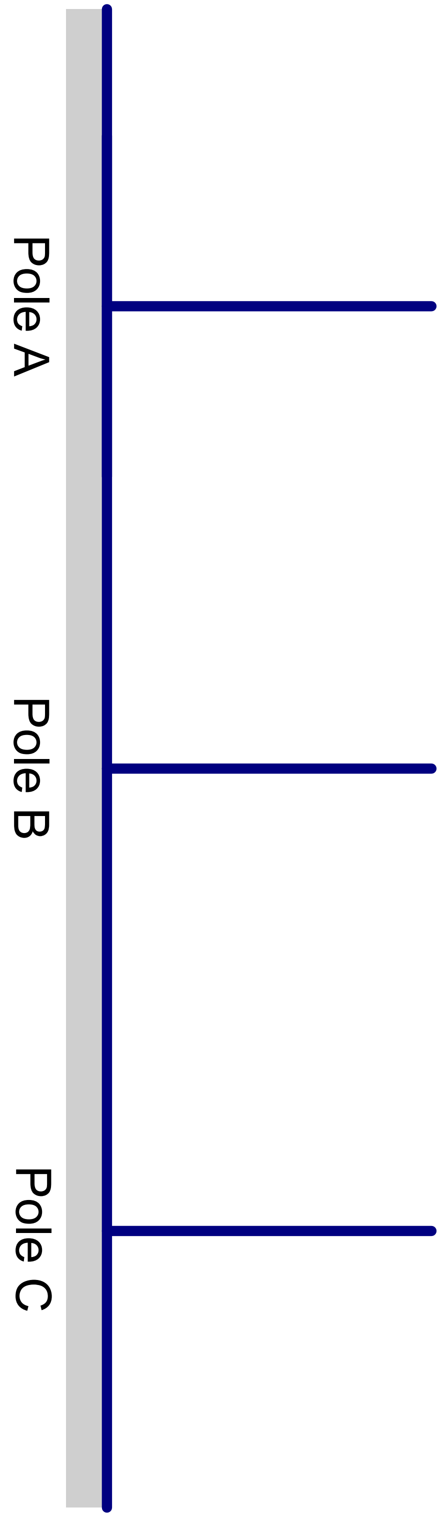
1. Given 3 disks in pole A, what is the minimum number of moves needed to move all of them to another pole?



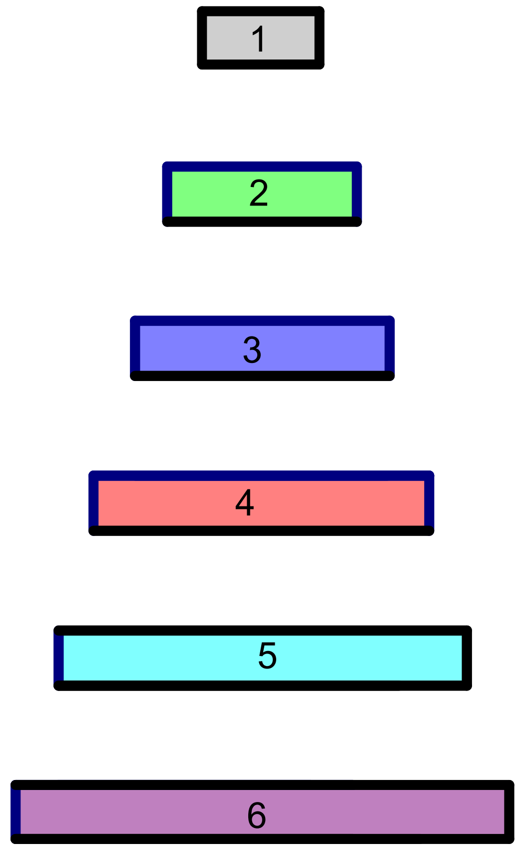
2. Compare your method with other members of your group. Did you take the same number of moves? How do you convince someone that you have the fewest possible number of moves?

3. Try the puzzle several more times, each time with a larger number of disks, what is the minimum number of moves needed for each new stack?

Resource: Tower of Hanoi Puzzle



Cut out these rectangles to be used as disks.



1.1.2 Generalize

Focus questions: How do we use the solution of one case to solve the next larger case?

1. Describe a way of working out how many moves are needed when one extra disk is added.

2. Explain why your method requires the minimum number of moves possible.

1.1.3 Reflect: Tower of Hanoi

The puzzle your class just solved is historically known as the Tower of Hanoi. According to legend, the priests of Brahma have been solving the Tower of Hanoi puzzle with 64 disks. When the last move of the puzzle is completed, the world will end.

1. Describe any results that you found in your investigation including:
   a. the results of any specific cases you tried,
   b. any patterns, procedures, or formulas that you uncovered, and why they work.

2. What is the smallest case that you tried or could have tried? Why is it important to consider the solution for the smallest possible case?

3. If you know the number of moves required for a certain number of disks (say 3 disks), how can you use this information to figure out the number of moves needed for a stack with one more disk (4 disks)? What is it about the rules of the puzzle that cause this to be true?

1.1.4 Investigate: Explicitly Speaking

Focus questions: Given any number of disks, what is the minimum number of moves required to complete the puzzle? What is it about the structure that makes it true?

1. According to the legend, the priests of Brahma have been solving the Tower of Hanoi puzzle with 64 disks. When the last move of the puzzle is completed, the world will end. How many moves will it take the priests to solve the puzzle?

2. Ryan carefully examined the pattern in the table that he built to solve the Tower of Hanoi puzzle. He claimed that the number of moves is equal to $2^n - 1$, where n is the number of disks.
   a. Verify whether Ryan's conjecture is true or not for the cases of up to 5 disks. Show your work in an organized manner.

   b. While working out part (a), Fernando commented that if Ryan's conjecture works for the cases of up to 5 disks, then it will work for any number of disks. How many cases would it take to convince you? Explain.

   c. Do you think Ryan's formula is true for any number of disks? If so, what makes it true? If not, find a counterexample.

1.1.5 Reflect

While solving the Tower of Hanoi puzzle, we hope that you have uncovered two formulas that can be used to calculate the number of moves for any stage. Reexamine the results and the thinking processes leading to those results.

1. Restate the formulas in your own words. Include any restrictions that should be imposed on the formulas.
2. How did you justify these formulas?
3. How are the formulas the same? How are they different?
4. What does one formula tell you while the other doesn't?

1.1.6 Extend: Don't Prove by Examples

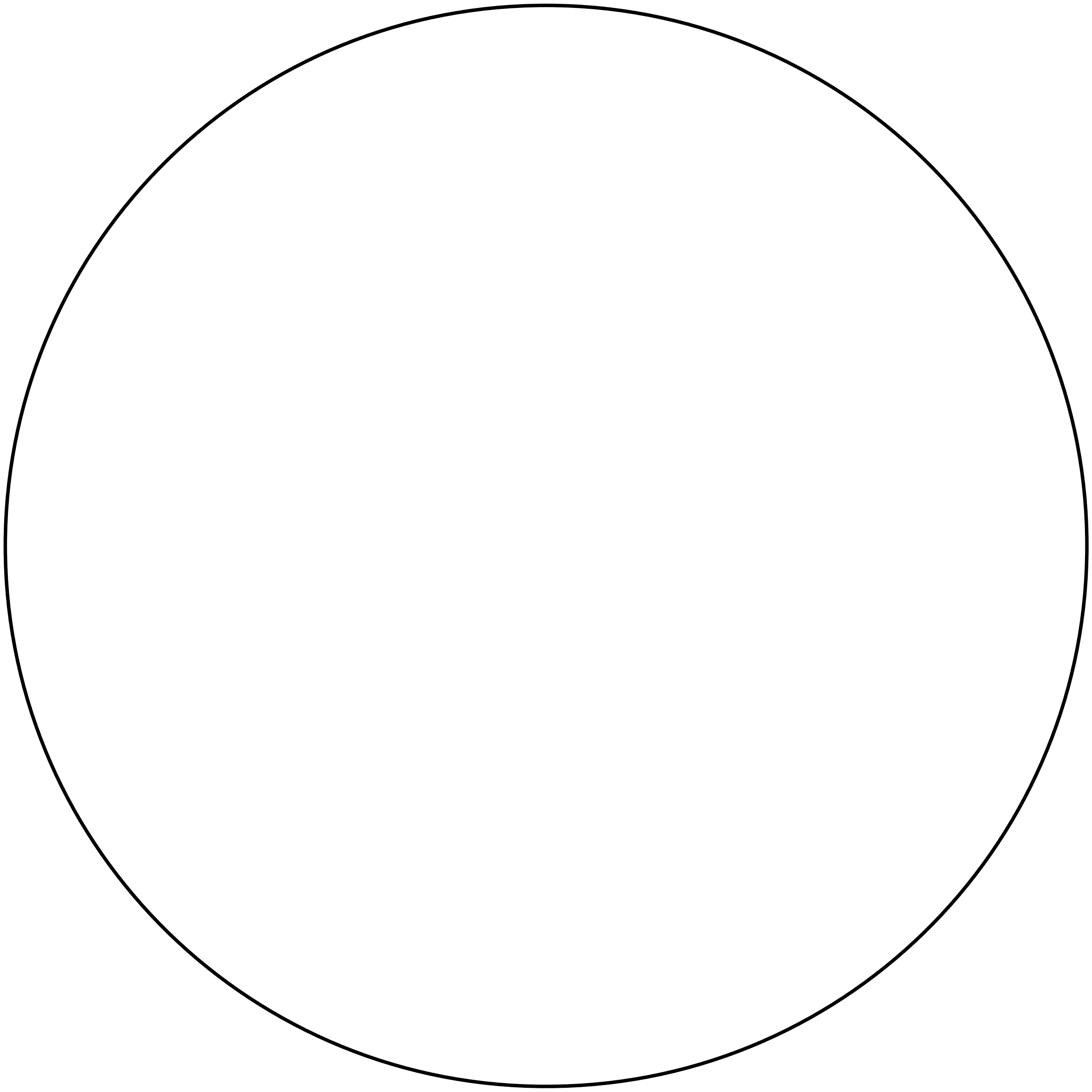
Focus question: How many examples are sufficient to justify a conjecture?

In the following task, you are going to place points on the given circle. Each time you add a new point, connect every pair of these points with a straight line so that no three lines cross at a point, and determine the number of regions created by these lines inside the circle. Complete the following steps.

- Place 1 point anywhere on the circle. How many regions are inside the circle?
- Place another point on the circle. Connect the two points with a straight line. How many regions are inside the circle now?
- Place a third point on the circle. Connect every pair of points with a straight line so that **no three lines cross at a point inside the circle**. How many regions are inside the circle now?
- Place a fourth point on the circle. Connect every pair of points with a straight line so that **no three lines cross at a point inside the circle**. How many regions are inside the circle now?
- Predict the number of regions inside the circle if there are 5 points.
  Now place a fifth point and connect every pair of points with a straight line so that **no three lines cross at a point inside the circle**. How many regions are inside the circle now? Is your prediction correct?
- Predict the number of regions inside the circle if there are 6 points.
  Now connect every pair of points with a straight line so that **no three lines cross at a point inside the circle**. How many regions are inside the circle now? Is your prediction correct?

Number of points on the circle	Number of regions inside circle

For use with 1.1.6: Don't Prove by Examples



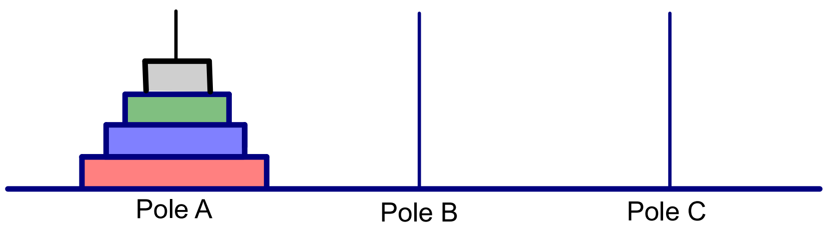
The Legend of the Golden Disks – Problem Set

Check for Understanding

1. What is the smallest case in the Tower of Hanoi puzzle? Why is it important to know the solution for the smallest case?

2. Describe the process that it takes to complete the Tower of Hanoi puzzle is you start with 4 disks. How many moves are required at the minimum? Remember the following rules:
   - Only one disk may be moved at a time.
   - A larger disk may not be put on top of a smaller disk.

   Your goal is to transfer all four disks from pole A to another pole.



3. Suppose you know the minimum number of moves required to move four disks, how can you use this information to figure out the number of moves needed for five disks? How many moves are required for 5 disks?

4. Suppose you know the minimum number of moves required to move 5 disks, how can you use this information to figure out the number of moves needed for 6 disks? How many moves are required for 6 disks?

Repeated Reasoning

5. Suppose you know the minimum number of moves required to move 20 disks, how can you use this information to figure out the number of moves needed for 21 disks?

6. Suppose you know the minimum number of moves required to move a certain number of disks, how can you use this information to figure out the total number of moves needed when another disk is added?

7. Naomi patiently solved the Tower of Hanoi puzzle in 513 moves. She claimed that this is the minimum number of moves possible for a certain number of disks. Is this possible? If so, how many disks did she use? If not, explain why not.

8. Paul also claimed that he solved the Tower of Hanoi puzzle in 2,047 moves. He claimed that this is the minimum number of moves possible for a certain number of disks. Is this possible? If so, how many disks did he use? If not, explain why not.

9. According to the legend, the priests of Brahma have been solving the Tower of Hanoi puzzle with 64 disks. When the last move of the puzzle is completed, the world will end. If one move is made each second, when will the world end?

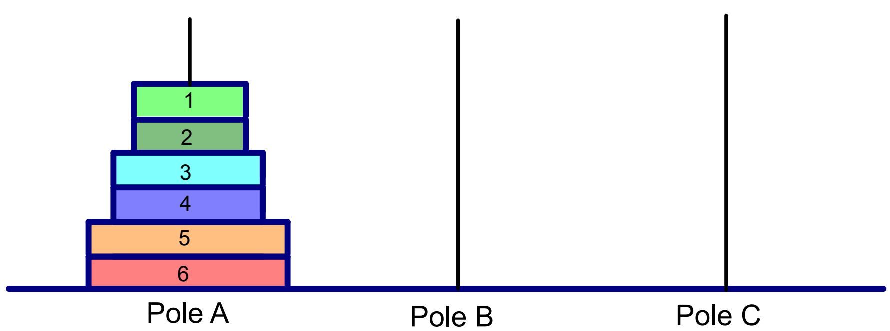
Diving Deeper

10. Your friend challenged you to solve a more complicated Tower of Hanoi puzzle. In this challenge, all the rules are the same as the original Tower of Hanoi puzzle except that you have two identical copies of each size.

    Remember that the goal is to move the stack of disks from the left pole (A) to the right pole (C) with the following conditions:

    - Only one disk may be moved at a time.
    - A larger disk may not be put on top of a smaller disk. (Disks may be placed on top of one of the same size.)

    a. Given 6 disks in pole A, two of each size, what is the minimum number of moves needed to move all of them to pole C?



    b. What is the minimum number of moves needed to move *2n* disks, two of each size? Explain.

11. Create a new rule or rules for the Tower of Hanoi puzzle. Describe the new rule(s) and a solution.

12. Counting Game: (From A Model for Reasoning with Recursion and Mathematical Induction in School Mathematics, PCMI 2010)

    Play the following game against another class member. Here are the rules.

    - At the start of the game, two players decide on a target number.
    - Starting from 1, the first player counts either one or two consecutive numbers. The second player continues counting 1 or 2 consecutive numbers after the last number called out by the first player. Take turn counting in this manner. Whoever reaches the target number is declared the winner.

      For example, player A and player B decide that the target number is 20.

      - Player A can start by saying "1" or "1, 2".
      - If player A says "1", then player B can say "2" or "2, 3".
      - If player A says "1, 2", then player B can say "3" or "3, 4".
      - The player that reaches "20" first is the winner.

    a. Play the game with another person several times. Determine the winning strategies for this game.

    b. Play the games several more times with different target numbers but keep the same rules. Answer the following questions.

       - Assuming that both players make the correct move on every turn, which player is guaranteed to win? Why?
       - How would your strategies change if we change the target number and keep the same rules?

    c. Suppose that a player can use 1, 2, or 3 consecutive numbers. How does this new rule affect your strategies?

    d. Generalize your strategies for any target number and any number of consecutive numbers that a player can use.